Lecture #16

CS 170

Spring 2021

Zero Sum Games

Ex: Rock-Paper-Scissors (R-P-S)

2 Players (call them Row and Col)

Both pick one of R, P, or S; who wins?

Row: \ Col	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

each enty of "utility matrix" says how much Row wins, = how much Col loses (reason for name "Zero Sum")

Row wants to maximize value of entry

Col wants to minimize value of entry

What is best strategy for Row? for Col?

What does "play the game" mean? (1/3)

$$\text{Row:}\quad \begin{array}{c|ccc} & \multicolumn{3}{c}{\text{Col}} \\ & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array}$$

$\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix}$ wants to $\begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix}$ entry

1) Row picks R, P or S, tells Col, then Col picks

Col always wins

2) Col picks R, P or S, tells Row, then Row picks

Row always wins

3) Row and Col pick, then announce at same time

What is best strategy for Row, or Col?

What does "play the game" mean? (2/3)

$$\text{Row: } \begin{array}{c|ccc} & \multicolumn{3}{c}{\text{Col}} \\ & R & P & S \\ \hline R & 0 & -1 & 1 \\ P & 1 & 0 & -1 \\ S & -1 & 1 & 0 \end{array}$$

$\begin{bmatrix} \text{Row} \\ \text{Col} \end{bmatrix}$ wants to $\begin{bmatrix} \text{maximize} \\ \text{minimize} \end{bmatrix}$ entry

3) Row and Col pick, then announce at same time

3a) Row (almost) always picks same row

Col notices this, can (almost) always win

3b) Col (almost) always picks same column

Row notices this, can (almost) always win

What does "play the game" mean? (3/3)

Col

$$\text{Row}:\ \begin{array}{cc|ccc|c} & & j=1 & j=2 & j=3 & \\ & & R & P & S & \\ \hline i=1 & R & 0 & -1 & 1 & x_1 \\ i=2 & P & 1 & 0 & -1 & x_2 \\ i=3 & S & -1 & 1 & 0 & x_3 \\ \hline & & y_1 & y_2 & y_3 & \end{array}$$

$\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix}$ wants to $\begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix}$ entry

3) Row and Col pick, then announce at same time

Row picks row i with probability x_i

Col picks column j with probability y_j

Let
$$U = \begin{array}{cc} & \begin{array}{ccc} & 2 & \end{array} \\ 2 & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}\end{array} = \text{"utility"}$$

Probability of picking $U(i,j) = x_i \cdot y_j$

Expected utility $= EU = \sum_{i,j} U(i,j) \cdot x_i \cdot y_j$

How should $\begin{bmatrix}\text{Row}\\ \text{Col}\end{bmatrix}$ pick $\begin{bmatrix}x_i\\ y_j\end{bmatrix}$ to $\begin{bmatrix}\text{maximize}\\ \text{minimize}\end{bmatrix}$ EU

Choosing a "strategy" for game

- Row's strategy $= x = (x_1, x_2, x_3)$
  Col's strategy $= y = (y_1, y_2, y_3)$

$$U = \begin{array}{cccc} R & P & S & \\ \left[\begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array}\right. & & \left.\right] & \begin{array}{c} R \; x_1 \\ P \; x_2 \\ S \; x_3 \end{array} \\ y_1 & y_2 & y_3 & \end{array}$$

- Ex: $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $y = (0, 0, 1)$

$$\mathbb{E}U = \sum_{i,j} U(i,j)\, x_i \cdot y_j = \frac{1}{3} \sum_i U(i,3) = 0$$

- Ex: $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $y = (y_1, y_2, y_3)$

$$\mathbb{E}U = \sum_{i,j} U(i,j)\, x_i y_j = \frac{1}{3} \sum_j y_j \sum_i U(i,j) = \frac{1}{3} \sum_j y_j \cdot 0 = 0$$

- Similarly, if $x = (x_1, x_2, x_3)$, $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ then $\mathbb{E}U = 0$
- Both Row and Col have strategies that guarantee the other player wins 0 in expectation
- Value of game $= \mathbb{E}U = 0$, with solution $(x,y) = \left(\begin{array}{c}\frac{1}{3}\frac{1}{3}\frac{1}{3} \\ \frac{1}{3}\frac{1}{3}\frac{1}{3}\end{array}\right)$

Let's try a different game

$$
\text{Row}\quad
\begin{array}{c|c|c|}
 & L & R \\
\hline
T & 5 & -3 \\
\hline
B & -1 & 1 \\
\hline
\end{array}
$$

$$\text{Col}$$

- Is $x = (\frac{1}{2}, \frac{1}{2})$ still a good strategy for Row?

  if Col picks $\begin{bmatrix} L \\ R \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \frac{1}{2}\cdot 5 + \frac{1}{2}(-1) = 2 \\ \frac{1}{2}(-3) + \frac{1}{2}(1) = -1 \end{bmatrix}$

  $\Rightarrow$ Col should choose R, loses $\mathbb{E}U = -1$, wins $+1$

- Is $y = (\frac{1}{2}, \frac{1}{2})$ still a good strategy for Col?

  if Row picks $\begin{bmatrix} T \\ B \end{bmatrix}$ then $\mathbb{E}U = \begin{bmatrix} \frac{1}{2}\cdot 5 + \frac{1}{2}(-3) = 1 \\ \frac{1}{2}\cdot(-1) + \frac{1}{2}(1) = 0 \end{bmatrix}$

  $\Rightarrow$ Row should choose T, wins $\mathbb{E}U = 1$

- Not like RPS, expect a better strategy

Let's try a different game

$$\text{Row}\quad \begin{array}{c c|c|c|} & & \frac{2}{5} & \frac{3}{5} \\ & & L & R \\ \hline \frac{1}{5} & T & 5 & -3 \\ \hline \frac{4}{5} & B & -1 & 1 \\ \hline \end{array}$$

$$\text{Col}$$

Are there better strategies for Row and Col than $\left(\frac{1}{2},\frac{1}{2}\right)$?

- Suppose Row plays $x=\left(\frac{1}{5},\frac{4}{5}\right)$

  if Col picks $\begin{bmatrix} L \\ R \end{bmatrix}$ then $\mathbb{E}U=\begin{bmatrix} \frac{1}{5}\cdot 5+\frac{4}{5}(-1)=\frac{1}{5} \\ \frac{1}{5}(-3)+\frac{4}{5}(1)=\frac{1}{5} \end{bmatrix}$

  Row wins $\frac{1}{5}$, no matter what Col's strategy is

- Suppose Col plays $y=\left(\frac{2}{5},\frac{3}{5}\right)$

  if Row picks $\begin{bmatrix} T \\ B \end{bmatrix}$ then $\mathbb{E}U=\begin{bmatrix} \frac{2}{5}\cdot 5+\frac{3}{5}(-3)=\frac{1}{5} \\ \frac{2}{5}(-1)+\frac{3}{5}(1)=\frac{1}{5} \end{bmatrix}$

  Col loses $\frac{1}{5}$ no matter what Row's strategy is

- Value of game $=\mathbb{E}U=\frac{1}{5}$ with solution (x,y)

Solving a Zero Sum Game as a LP (1/2)

		L	R
Row	x_1 T	5	-3
	x_2 B	-1	1

Col

Rephrase finding the best strategies for Row and Col as dual LPs

- Row's goal: choose $x=(x_1,x_2)$ to maximize payoff from Col's best response:

  pick x to maximize $\min(\overset{L}{5x_1-x_2}, \overset{R}{-3x_1+x_2})$

- Convert to LP: for Row

  constraints: $x_1 \geq 0$, $x_2 \geq 0$, $x_1+x_2=1$

  compute min(") maximize z : $z \leq 5x_1-x_2$, $z \leq -3x_1+x_2$

  LP: maximize $z = 1\cdot z + 0\cdot x_1 + 0 x_2$

  s.t. $x_1 \geq 0$, $x_2 \geq 0$, $x_1+x_2=1$, $z \leq 5x_1-x_2$, $z \leq -3x_1+x_2$

Solving a Zero Sum Game as a LP (2/2)

$$\begin{array}{ccc} & y_1 & y_2 \\ & L & R \\ \text{Row} \quad T & 5 & -3 \\ B & -1 & 1 \end{array}$$

Col

Rephrase finding the best strategies for Row and Col as dual LPs

- Col's goal: choose $y=(y_1, y_2)$ to minimize payoff from Row's best response: pick y to minimize $\max(\overset{T}{5y_1-3y_2}, \overset{B}{-y_1+y_2})$
- Convert to LP:
  constraints: $y_1 \geq 0$, $y_2 \geq 0$, $y_1+y_2=1$
  compute max(") minimize w: $w \geq 5y_1-3y_2$, $w \geq -y_1+y_2$
  LP: minimize $w = 1\cdot w + 0\cdot y_1 + 0\cdot y_2$
  s.t. $y_1 \geq 0$, $y_2 \geq 0$, $y_1+y_2=1$, $w \geq 5y_1-3y_2$, $w \geq -y_1+y_2$

These Two LPs are Dual

- Row's LP: maximize

$$z = 1\cdot z + 0\cdot x_1 + 0\cdot x_2 \quad \text{s.t.}$$

$$x_1 \ge 0,\ x_2 \ge 0,\ x_1 + x_2 \underset{\ge}{\overset{}{\underset{\le}{=}}} 1,$$

$$z \le 5x_1 - x_2,\ z \le -3x_1 + x_2$$

- Write $z = z_1 - z_2,\ z_1 \ge 0,\ z_2 \ge 0$

$$\max z = \underbrace{[1, -1, 0, 0]}_{c^T} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix}}_{\tilde{x}} = c^T \tilde{x}$$

s.t.

$$\begin{aligned} z_1 - z_2 - 5x_1 + x_2 &\le 0 \\ z_1 - z_2 + 3x_1 - x_2 &\le 0 \\ x_1 + x_2 &\le 1 \\ -x_1 - x_2 &\le -1 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & -5 & 1 \\ 1 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\max c^T \tilde{x} \quad \text{s.t.} \quad A \cdot \tilde{x} \le b, \quad \tilde{x} \ge 0$$

- Col's LP: minimize

$$w = 1\cdot w + 0\cdot y_1 + 0\cdot y_2 \quad \text{s.t.}$$

$$y_1 \ge 0,\ y_2 \ge 0,\ y_1 + y_2 = 1$$

$$w \ge 5y_1 - 3y_2,\ w \ge -y_1 + y_2$$

- Write $w = w_1 - w_2,\ w_1 \ge 0,\ w_2 \ge 0$

$$\min w = \underbrace{[0, 0, 1, -1]}_{b^T} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix}}_{\tilde{y}} = b^T y$$

s.t.

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -5 & 3 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{A^T} \cdot \begin{bmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{bmatrix} \ge \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\min b^T \tilde{y} \quad \text{s.t.} \quad A^T \tilde{y} \ge c, \quad \tilde{y} \ge 0$$

Dual $\Rightarrow$ Row's and Col's strategies match: $z = c^T \tilde{x} = b^T \tilde{y} = w$

General Approach to Zero Sum Games

- Game specified by $m \times n$ matrix U

  Row strategy $= x = (x_1, x_2, \dots x_m)^T$, $x_i \ge 0$, $\sum_i x_i = 1$

  Col strategy $= y = (y_1, y_2, \dots, y_n)^T$, $y_j \ge 0$, $\sum_j y_j = 1$

- Expected Utility $= \mathbb{E}U = \sum_{i,j} U(i,j) \cdot x_i \cdot y_j = x^T U y$

Row's Goal:

pick x to maximize $\min(x^T U)$

maximize z : $z \le (x^T U)_j$ $j = 1:n$

$x_i \ge 0$ $i = 1:m$ $\sum_i x_i = 1$

Col's Goal:

pick y to minimize $\max(Uy)$

minimize w

$w \ge (Uy)_i$ $i = 1:m$

$y_j \ge 0$, $j = 1:n$ $\sum y_j = 1$

Dual: $\max z = \min w =$ Value of Game

Solve using simplex...

Game Theory - A little history

- Von Neumann — before duality of LPs
- Many variations:
  - more than 2 players
  - nonzero/zero sum
  - perfect/imperfect information (poker)
  - combinatorial/non combinatorial (chess)
  - discrete vs continuous (robot motion planning)
  - ...
- Many Applications
  - economics, computational complexity, sociology, biology, philosophy...
- Many Prizes
  - 5 Nobel Prizes (11 recipients) Economics
  - 4 Oscars — A Beautiful Mind — John Nash